SPLITTING BASED ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

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Outline

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Motivation

Example: There are 27 pieces of fruit in a barrel, and twice as many oranges as apples. How many apples and oranges are in the barrel?

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$$x + y = 27$$

$$y = 2x$$

Note: There are may problems which can be modelled as linear system equations.

System of Equations

- A system of linear equations will be of the form

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

- We can write in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

- So the general form $Ax = b$

Elementary Row Operations

The following operations applied to any matrix, yields a rwo-equivalent form.

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- Interchanges: The order of two rows can be changed ($R_i \leftrightarrow R_j$).
- Scaling: Multiplying a row by a nonzero constant ($R_i \rightarrow cR_i$).
- Replacement: The row can be replaced by the sum of that row and a nonzero multiple of any other row ($R_i \rightarrow R_i + cR_j$).

An inconsistent example

Consider the following linear system

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

System of Linear Equations

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Clearly this system of equations is not solvable or inconsistent. (Why?)

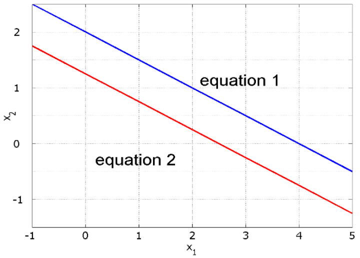

FIGURE: No solutions

No Solutions: If $\text{rank}(A) \neq \text{rank}([A|b])$.

Uniqueness of solutions

Consider the following linear system

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

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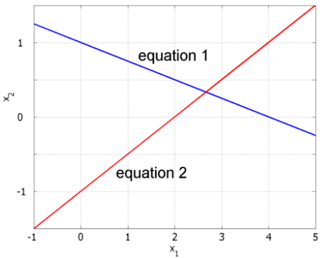

FIGURE: Unique Solution

The above system has unique solution $x_1 = 8/3$ and $x_2 = 2/3$.

Unique Solution: If rank(A) =rank($[A|b]$) = No of variables.

Rank deficient matrices/Infinite number of solutions

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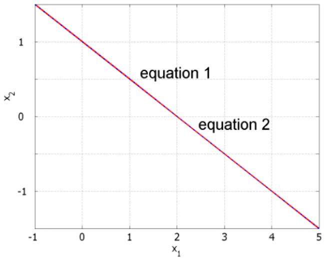

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Solution Techniques

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Direct Methods:

- Find a solution in a finite number of operations by transforming the system into an equivalent system that is 'easier' to solve.
- Diagonal, upper or lower triangular systems are easier to solve.
- Number of operations is a function of system size n.

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Iterative Methods:

- Computes successive approximations of the solution vector x for a given A and b, starting from an initial point x_0.
- Total number of operations is uncertain, may not converge.

Gaussian Elimination

Consider a system $Ax = b$. By using elementary row operations, the augmented matrix $[A|b]$ is transformed into an upper triangular matrix (all elements below the pivot element are 0)

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Row Echelon Form: After applying forward elimination, the augmented matrix will be in the following row echelon form:

$$\left[\begin{array}{cccc|c} a'_{11} & a'_{12} & \cdots & a'_{1n} & b'_1 \\ 0 & a'_{22} & \cdots & a'_{2n} & b'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a'_{mn} & b'_m \end{array}\right]$$

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Back Substitution: Solve for x_i:

$$x_i = \frac{b'_i - \sum_{j=i+1}^{n} a'_{ij} x_j}{a'_{ii}}$$

Drawbacks of Gaussian Elimination

Recall:

Example: Solve the following system

$$\begin{aligned} 1.133x_1 + 5.281x_2 &= 6.414 \\ 24.14x_1 - 1.210x_2 &= 22.93 \end{aligned}$$

DRAWBACKS OF GAUSSIAN ELIMINATION

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$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array}\right]$$

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$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & -113.8 \end{array}\right]$$

Hence $x_1 = 0.9956,\ x_2 = 1.001$ but the exact solution is $x_1 = 1,\ x_2 = 1$.

- When the pivotal element is very small, the multipliers will be large.

Pivoting Strategies (Partial Pivoting)

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- When the pivotal element is very small, the multipliers will be large.
- Adding numbers of widely different magnitudes can lead to a loss of significance.
- To reduce error, row interchanges are made to maximize the magnitude of the pivot element.

Partial Pivoting

Forward Elimination: After applying forward elimination, the augmented matrix at step$-i$ will be in the following row echelon form:

$$\left[\begin{array}{cccccccc|c}
a_{11}^{(i)} & a_{12}^{(i)} & \cdots & a_{1i}^{(i)} & \cdots & a_{1j}^{(i)} & \cdots & a_{1n}^{(i)} & b_1^{(i)} \\
0 & a_{22}^{(i)} & \cdots & a_{2i}^{(i)} & \cdots & a_{2j}^{(i)} & \cdots & a_{2n}^{(i)} & b_2^{(i)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a_{ii}^{(i)} & \cdots & a_{ij}^{(i)} & \cdots & a_{in}^{(i)} & b_i^{(i)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a_{ji}^{(i)} & \cdots & a_{jj}^{(i)} & \cdots & a_{jn}^{(i)} & b_j^{(i)} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & a_{mi}^{(i)} & \cdots & a_{mj}^{(i)} & \cdots & a_{mn}^{(i)} & b_m^{(i)}
\end{array}\right]$$

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0 & 0 & \cdots & a_{mi}^{(i)} & \cdots & a_{mj}^{(i)} & \cdots & a_{mn}^{(i)} & b_m^{(i)}
\end{array}\right]$$

If $\max\{|a_{ii}^{(i)}|, \ |a_{i+1\,i}^{(i)}|, \ldots, \ |a_{ji}^{(i)}|, \ldots, \ |a_{mi}^{(i)}|\} = |a_{ji}^{(i)}| \neq 0$ then swap row i with row j.

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$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right]$$

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$$\left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right]$$

Hence $x_1 = 1, \; x_2 = 1$.

DRAWBACKS OF PARTIAL PIVOTING

Example: Solve the following system

$$\begin{aligned} 30x_1 + 591400x_2 &= 591700 \\ 5.291x_1 - 6.130x_2 &= 46.78 \end{aligned}$$

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$$\|A\|_\infty = \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|,\ \|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}|,\ \|A\|_2 = \sqrt{\lambda_{\max}(A^*A)},\ \ \|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{trace(A^*A)}.$$

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- Condition number: $k(A) = \|A\|\|A^{-1}\|$.
- Effect of condition number:
- If the coefficient matrix is ill-conditioned then round-off will lead huge error in the solution.
- if A is ill-conditioned (a small change in some entries leads nonsingular to singular), A^{-1} will not be computed accurately.

Disasters due to bad numerics

On February 25, 1991, during the Gulf War, an American Patriot Missile battery in Dharan, Saudi Arabia, failed to track and intercept an incoming Iraqi Scud missile. This resulted in 28 deaths and 100 injuries.

DRAWBACKS OF GAUSSIAN ELIMINATION

Disasters due to bad numerics

On June 4, 1996 an unmanned Ariane 5 rocket launched by the European Space Agency exploded just 40seconds after its lift-off from Kourou, French Guiana. Ariane explosion costing \$7 billion + The destroyed rocket and its cargo were valued at \$500 million.

The rocket exploded seconds after launching

and so on ....

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Iterative Methods for solving Linear Systems

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- For systems of small dimension, there is no need. Direct techniques will perform very efficiently.
- For systems with large, sparse coefficient matrices, direct techniques are often less efficient than iterative techniques.
- More appropriate when the number of equations involved is large, or when the matrix is sparse (many coefficients whose value is zero).

ITERATIVE SOLUTION PROCEDURE

Main Steps:

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Iterative methods:

- Jacobi's Method (Carl Gustav Jakob Jacobi, 1804-1851)
- Gauss-Seidel Method (Carl Friedrich Gauss 1777-1855, Philipp Ludwig von Seidel 1821-1896)
- Successive Overrelaxation (SOR) Method

Jacobi's Method

Consider the linear system:

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

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The Jacobi Method is derived by decomposing the matrix A into its diagonal (D), strictly lower triangular (L) and strictly upper triangular (U) such that

$$A = D + L + U.$$

- Thus the system $Ax = b$ is rewritten as

$$x = -D^{-1}(L+U)x + D^{-1}b = Tx + c,$$

where $T = -D^{-1}(L+U) = D^{-1}(D-A)$ and $c = D^{-1}b$

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- Hence the Jacobi's iterative method in matrix form is given by

$$x^{(k)} = Tx^{(k-1)} + c, \quad \text{where } T = -D^{-1}(L+U), \; c = D^{-1}b$$

and $k = 1, 2, 3, \ldots$

- Jacobi's iterative method in component form is given by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[b_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j^{(k-1)} \right], \quad i = 1, 2, \ldots, n \text{ and } k = 1, 2, 3, \ldots$$

Jacobi's Method

Example: Use 5-digit rounding and Jacobi method to solve the following system:

$$\begin{aligned} 2x_1 - x_2 &= 1 \\ -x_1 + 3x_2 - x_3 &= 8 \\ -x_2 + 2x_3 &= -5 \end{aligned}$$

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Solution: From $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $(L+U) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}$,

we have

$$T = -D^{-1}(L+U) = \begin{bmatrix} 0 & 0.5000 & 0 \\ 0.3333 & 0 & 0.3333 \\ 0 & 0.5000 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0.5000 \\ 2.6667 \\ -2.5000 \end{bmatrix}$$

Jacobi's Method

Iteration-1: Taking $x^{(0)} = [0\ 0\ 0]^T$, we can compute

$$x^{(1)} = Tx^{(0)} + c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5000 \\ 2.6667 \\ -2.5000 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 2.6667 \\ -2.5000 \end{bmatrix}$$

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Iteration-2:

$$x^{(2)} = \begin{bmatrix} 0 & 0.5000 & 0 \\ 0.3333 & 0 & 0.3333 \\ 0 & 0.5000 & 0 \end{bmatrix} \begin{bmatrix} 0.5000 \\ 2.6667 \\ -2.5000 \end{bmatrix} + \begin{bmatrix} 0.5000 \\ 2.6667 \\ -2.5000 \end{bmatrix} = \begin{bmatrix} 1.8333 \\ 2.0000 \\ -1.1667 \end{bmatrix}$$

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After 20 iterations (Iteration-21):

$$x^{(21)} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \Leftrightarrow x_1 = 2,\ x_2 = 3,\ x_3 = -1$$

Gauss-Seidel Method

- In Gauss-Seidel method, we rewrite the system $Ax = b$ as

$$x = -(D+L)^{-1}Ux + (D+L)^{-1}b = Tx + c,$$

where $T = -(D+L)^{-1}U$ and $c = (D+L)^{-1}b$

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- Hence the Gauss-Seidel iterative method in matrix form is given by

$$x^{(k)} = Tx^{(k-1)} + c, \quad \text{where } T = -(D+L)^{-1}U, \; c = (D+L)^{-1}b$$

and $k = 1, 2, 3, \ldots$

- Jacobi's iterative method in component form is given by

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right],$$

where $i = 1, 2, \ldots, n$ and $k = 1, 2, 3, \ldots$

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Solution: From $D + L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 2 \end{bmatrix}$, $U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}$,

we have

$$T = -(D+L)^{-1}U = \begin{bmatrix} 0 & 0.5000 & 0 \\ 0 & 0.1667 & 0.3333 \\ 0 & 0.0833 & 0.1667 \end{bmatrix}, \quad c = \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix}$$

Gauss-Seidel Method

Iteration-1: Taking $x^{(0)} = [0\ 0\ 0]^T$, we can compute

$$x^{(1)} = Tx^{(0)} + c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix}$$

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Iteration-2:

$$x^{(2)} = \begin{bmatrix} 0 & 0.5000 & 0 \\ 0 & 0.1667 & 0.3333 \\ 0 & 0.0833 & 0.1667 \end{bmatrix} \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix} + \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix} = \begin{bmatrix} 1.9167 \\ 2.9444 \\ -1.0278 \end{bmatrix}$$

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Iteration-3:

$$x^{(3)} = \begin{bmatrix} 0 & 0.5000 & 0 \\ 0 & 0.1667 & 0.3333 \\ 0 & 0.0833 & 0.1667 \end{bmatrix} \begin{bmatrix} 1.9167 \\ 2.9444 \\ -1.0278 \end{bmatrix} + \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix} = \begin{bmatrix} 1.9722 \\ 2.9815 \\ -1.0093 \end{bmatrix}$$

Gauss-Seidel Method

Iteration-8:

$$x^{(8)} = \begin{bmatrix} 0 & 0.5000 & 0 \\ 0 & 0.1667 & 0.3333 \\ 0 & 0.0833 & 0.1667 \end{bmatrix} \begin{bmatrix} 1.9997 \\ 2.9998 \\ -1.0001 \end{bmatrix} + \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix} = \begin{bmatrix} 1.9999 \\ 2.9999 \\ -1.0000 \end{bmatrix}$$

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After 8 iterations (Iteration-9):

$$x^{(9)} = \begin{bmatrix} 0 & 0.5000 & 0 \\ 0 & 0.1667 & 0.3333 \\ 0 & 0.0833 & 0.1667 \end{bmatrix} \begin{bmatrix} 1.9999 \\ 2.9999 \\ -1.0000 \end{bmatrix} + \begin{bmatrix} 0.5000 \\ 2.8333 \\ -1.0833 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

Therefore, we obtain the solution $x_1 = 2, \; x_2 = 3, \; x_3 = -1$

Spectral radius and norm of a matrix

Recall:

- Norm of a matrix ($A_{n\times n}$):

$$\|A\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|a_{ij}|,\ \|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|,\ \|A\|_2 = \sqrt{\lambda_{\max}(A^*A)},\ \ \|A\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{trace(A^*A)}.$$

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- Spectral radius: The spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\rho(A)$ and defined by
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- Relation between norm and $\rho(A)$: $\rho(A) \le \|A\|$.

Stopping Criteria

The following stopping rules are commonly used.

- Stop if the successive error, $\|x^{(k)} - x^{(k-1)}\| < \epsilon$.
- Stop if the residual error, $\|b - Ax^{(k)}\| < \epsilon$.
- Stop if the relative error, $\dfrac{\|x^{(k)} - x^{(k-1)}\|}{\|x^{(k)}\|} < \epsilon$.

Convergence of iterative methods

Theorem

Consider a non-singular system $Ax = b$ and its equivalent form be $x = Tx + c$. Then for any $x^{(0)} \in \mathbb{R}^n$, the sequence $\{x^{(k)}\}$ defined by $x^{(k)} = Tx^{(k-1)} + c$ $(k = 1, 2, \ldots)$, converges to the unique solution $A^{-1}b$, of the system $Ax = b$ if and only if $\rho(T) < 1$.

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Corollary

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Convergence of Jacobi's and Gauss-Seidel Method

Theorem

If A is strictly diagonally dominant (or symmetric and positive definite), then for any choice $x^{(0)}$, the sequence $\{x^{(k)}\}$ obtained by both the Jacobi and Gauss-Seidel iterative methods, converge to the unique solution $A^{-1}b$, of $Ax = b$.

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Theorem (Stein-Rosenberg)

If $a_{ij} \leq 0$ for each $i \neq j$ and $a_{ii} > 0$ for each $i = 1, 2, \ldots, n$. Then one and only one of the following statements holds:

- $0 \leq \rho(T_G) < \rho(T_J) < 1$
- $1 < \rho(T_J) < \rho(T_G)$
- $\rho(T_J) = 0 = \rho(T_G)$
- $\rho(T_J) = 1 = \rho(T_G)$

Successive Overrelaxation (SOR) Method

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where T_ω is the iteration matrix.

- Choose ω such that $\rho(T_\omega) < 1$ or $\|T_\omega\| < 1$.

SOR Method based on Jacobi Method

- It extends the Jacobi method by introducing a relaxation factor ω to improve convergence.
- The iterative formula for the SOR method in matrix form, based on the Jacobi method, is:

$$x^{(k+1)} = (1 - \omega)x^{(k)} + \omega D^{-1}\left(b - (L + U)x^{(k)}\right)$$

where $A = D + L + U$ is the decomposition of matrix A into its diagonal (D), strictly lower triangular (L), and strictly upperr triangular (U).

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Component-wise: For $i = 1, 2, \ldots, n$,

$$x_i^{(k)} = x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{n} a_{ij}x_j^{(k-1)}\right], \quad k = 1, 2, 3, \ldots$$

SOR Iteration Formula for Gauss-Seidel Method

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Component-wise: For $i = 1, 2, \ldots, n$,

$$x_i^{(k)} = x_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i}^{n} a_{ij} x_j^{(k-1)} \right], \quad k = 1, 2, 3, \ldots$$

SOR Iteration Methods

Example: Solve the following system

$$4x_1 + 3x_2 = 24, \quad 3x_1 + 4x_2 - x_3 = 30, \quad -x_2 + 4x_3 = -24$$

- by Gauss-Seidel method with $x^{(0)} = (1, 1, 1)^T$
- by Gauss-Seidel with SOR and $\omega = 1.25$, $x^{(0)} = (1, 1, 1)^T$

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Solution: $T_{gs} = \begin{bmatrix} 0 & -0.7500 & 0 \\ 0 & 0.5625 & 0.2500 \\ 0 & 0.1406 & 0.0625 \end{bmatrix}, \; c_{gs} = \begin{bmatrix} 6.0000 \\ 3.0000 \\ -5.2500 \end{bmatrix}$

$$T_\omega = \begin{bmatrix} -0.2500 & -0.9375 & 0 \\ 0.2344 & 0.6289 & 0.3125 \\ 0.0732 & 0.1965 & -0.1523 \end{bmatrix}, \; c_\omega = \begin{bmatrix} 7.5000 \\ 2.3438 \\ -6.7676 \end{bmatrix}$$

SOR Iteration Methods

TABLE: Gauss-Seidel with SOR

k	1	2	3	4	5	6	7
$x^{(k)}$	6.3125 3.5195 -6.6501	2.6223 3.9585 -4.6004	3.1333 4.0103 -5.0967	2.9571 4.0075 -4.9735	3.0037 4.0029 -5.0057	2.9963 4.0009 -4.9983	3.0000 4.0003 -5.0003

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Convergence of SOR Iteration Methods

- The convergence of the SOR method depends on the choice of ω.

CONVERGENCE OF SOR ITERATION METHODS

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THEOREM (KAHAN)

Consider a system $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. If $a_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$ then $\rho(T_\omega) \geq |\omega - 1|$.

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Note: The above theorem tells us that the SOR method can converge only if $0 < \omega < 2$.

Theorem (Ostrowski-Reich)

Consider a system $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. If A is positive definite matrix and $0 < \omega < 2$, then the SOR method converges for any choice of $x^{(0)}$.

OPTIMUM VALUE OF ω FOR SOR ITERATION METHODS

The optimal value of ω minimizes the spectral radius of the iteration matrix.

THEOREM

Consider a system $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. If A is positive definite and tridiagonal, then

- $\rho(T_{gs}) = \rho(T_j) < 1$
- *the optimal choice for ω is*

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_j)]^2}},$$

  where T_j is the iteration matrix of Jacobi's Method.
- $\rho(T_\omega) = \omega - 1$

Under-relaxation: If $0 < \omega < 1$, the method is called under-relaxed.

- It may converge slowly.
- It is used to make a non-convergent system converge, or to speedup convergence by avoiding oscillations

Classification of SOR Methods

Under-relaxation: If $0 < \omega < 1$, the method is called under-relaxed.

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Over-relaxation: If $1 < \omega < 2$, the method is called over-relaxed.

- It generally converges faster.
- It is used to accelerate the convergence, if approximate solution moving toward the exact solution, at a slow rate.

Optimum value of ω for SOR Iteration Methods

Example: Find the optimal choice of ω for the following system

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- Compute eigenvalues of T_j : $\lambda_1 = 0, \ \lambda_2 = \sqrt{0.625}, \ \lambda_3 = -\sqrt{0.625}$.

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- Compute eigenvalues of T_j : $\lambda_1 = 0, \ \lambda_2 = \sqrt{0.625}, \ \lambda_3 = -\sqrt{0.625}$.
- Compute optimum value of ω :

$$\omega = \frac{2}{1 + \sqrt{1 - 0.625}} \approx 1.24$$

Application

Consider the two-dimensional Poisson's equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \blacksquare = [0, 1] \times [0, 1]$$

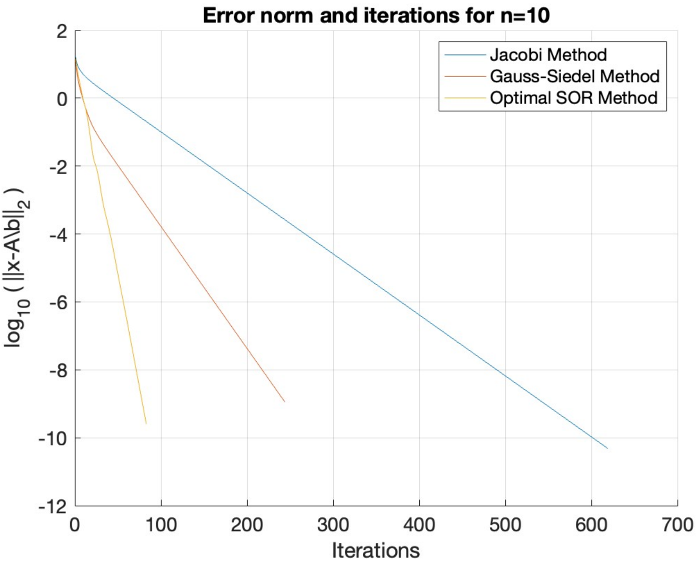
with $u(x, y) = 0$ on the boundary $\partial\blacksquare$. Using 5-point stencil central difference scheme on a discretizing the unit square domain with n interior nodes, we obtain the following system

$$Ax = b, \; A \in \mathbb{R}^{n^2 \times n^2}, \; b \in \mathbb{R}^{n^2}$$

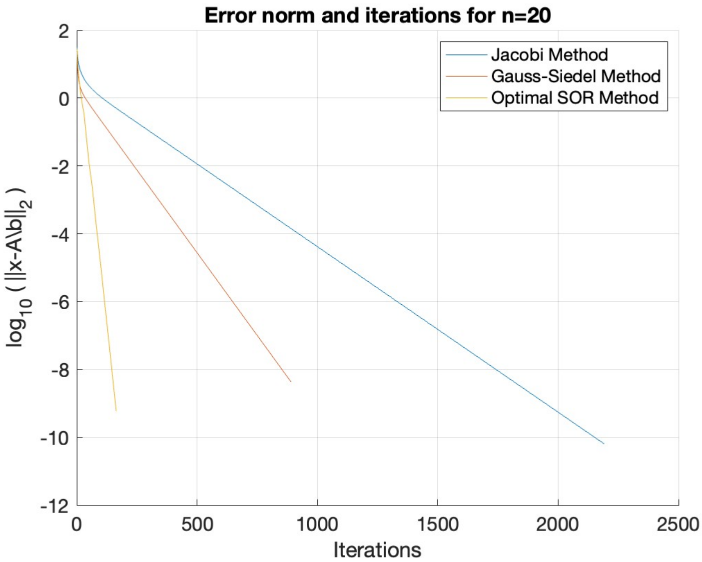
and the coefficient matrix will be of the form $A = I \otimes P + P \otimes I$, where

$$P = \begin{pmatrix} -2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & -2 \end{pmatrix}.$$

Application



APPLICATION



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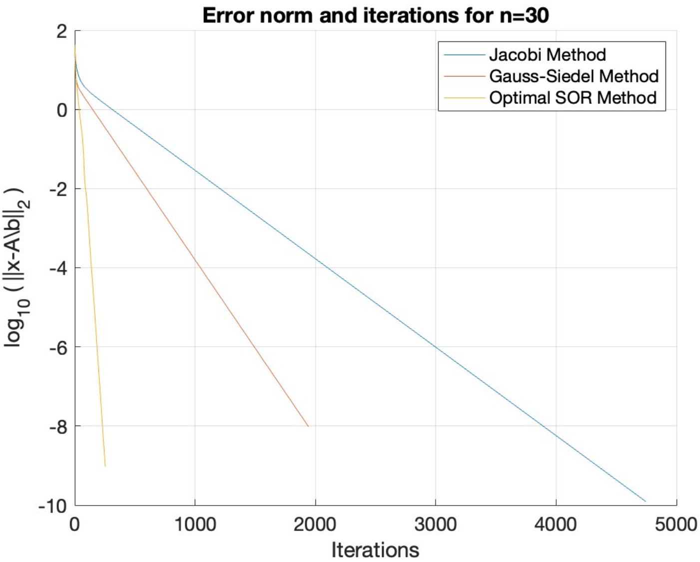

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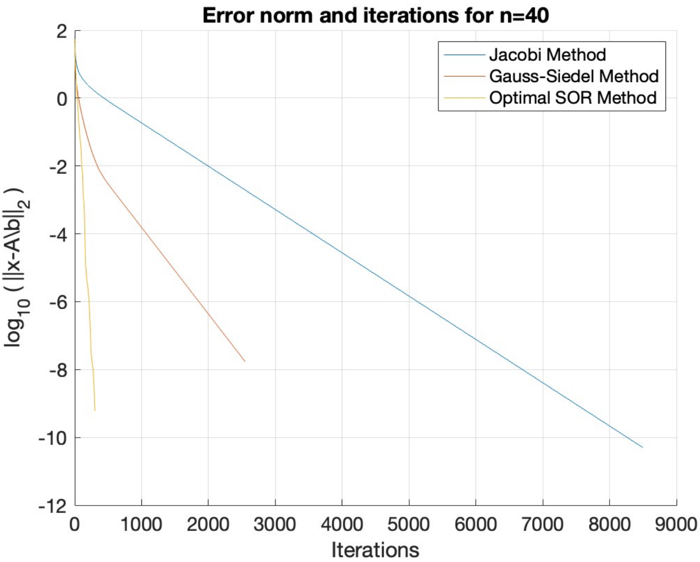

Outline

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MATRIX SPLITTING BASED ITERATIVE SCHEMES

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- To deal such systems, in recent past many researchers has considered the splitting theory such as proper regular splitting and proper weak regular splitting.For example, If $A = B - C$ is a proper splitting of $A \in \mathbb{R}^{m\times n}$, then the iterative scheme

$$x^{k+1} = B^{\dagger}Cx^{k} + B^{\dagger}b \qquad (1)$$

for (1) converges to $A^{\dagger}b$, the least squares solution for any initial vector x^0 iff $\rho(B^{\dagger}C) < 1$.

Drawbacks:

- In the previous iterative scheme, we need to check the spectral radius and computing spectral radius for large system is computationally expensive.

INTRODUCTION

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In this talk, we will discuss an alternating iterative scheme which can avoid the spectral radius calculation.

Basic Terminology

Definition (Moore-Penrose inverse)

Let $A \in \mathbb{C}^{m\times n}$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies the following properties $AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA$, then X is called the Moore-Penrose inverse of A and denoted as $A^\dagger$.

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Definition

A splitting $A = B - C$ is called proper splitting of $A \in \mathbb{R}^{m\times n}$ if $R(A) = R(B)$ and $N(A) = N(B)$.

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A splitting $A = B - C$ is called proper splitting of $A \in \mathbb{R}^{m \times n}$ if $R(A) = R(B)$ and $N(A) = N(B)$.

Definition (Definition 1.1, 1.2[5])

Let $A = B - C$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Then the splitting is called proper regular splitting if $B^\dagger \geq 0$ and $C \geq 0$ and called a proper weak regular splitting if $B^\dagger \geq 0$ and $B^\dagger C \geq 0$.

Basic Terminology

Definition (Moore-Penrose inverse)

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[5] L. Jena, D. Mishra, and S. Pani. Convergence and comparison theorems for single and double decompositions of rectangular matrices. Calcolo, 51(1):141–149, 2014.

Basic Terminology & Results

Based on the above definitions and splitting, the following results have been proved in [5] and [2].

Theorem (Theorem 1.3, [5])

Let $A = B - C$ be a proper regular splitting of $A \in \mathbb{R}^{m \times n}$. Then the Moore-Penrose inverse $A^\dagger \geq 0$ if and only if the spectral radius of the iteration matrix is less than 1, i.e., $\rho(B^\dagger C) < 1$.

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BASIC TERMINOLOGY & RESULTS

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[2] A. Berman and R. J. Plemmons. Cones and iterative methods for best least squares solutions of linear systems. SIAM Journal on Numerical Analysis, 11(1):145-154, 1974.

OUTLINE

Three step Alternating iterative Scheme

- Let $A = B - C = X - Y = S - T$ be three proper splittings of the matrix $A \in \mathbb{R}^{m \times n}$. The followings are the proposed iterative schemes for the above three splittings

$$x^{k+1/3} = B^{\dagger} C x^{k} + B^{\dagger} b \tag{2}$$

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which provide the solution of the system (1) iteratively for any initial guess x^0.

- By simplifying the iterative schemes (2), (3) and (4) we have the alternating iteration

$$x^{k+1} = Hx^{k} + Qb, \tag{5}$$

where $H = S^{\dagger} T X^{\dagger} Y B^{\dagger} C$ and $Q = S^{\dagger}(T X^{\dagger} Y B^{\dagger} + T X^{\dagger} + I)$.

Three step Alternating iterative Scheme

Note that convergence of individual splitting does not imply the convergence of alternating iterative scheme which can be seen in the next example.

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Example

Consider $A = \begin{bmatrix} 1 & 5 & -2 & -3 \\ 2 & -2 & 4 & -2 \\ -1 & 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{33}{10} & \frac{-21}{10} & \frac{9}{2} & \frac{-51}{5} \\ \frac{-11}{5} & \frac{32}{5} & \frac{-11}{2} & \frac{93}{10} \\ \frac{-23}{5} & \frac{77}{10} & \frac{-48}{5} & \frac{119}{10} \end{bmatrix} - \begin{bmatrix} \frac{23}{10} & \frac{-71}{10} & \frac{13}{2} & \frac{-36}{5} \\ \frac{-21}{5} & \frac{42}{5} & \frac{-19}{2} & \frac{113}{10} \\ \frac{-18}{5} & \frac{77}{10} & \frac{-43}{5} & \frac{89}{10} \end{bmatrix}$

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$$= \begin{bmatrix} \frac{46}{5} & \frac{29}{2} & \frac{-87}{10} & \frac{-368}{5} \\ \frac{2}{5} & \frac{-47}{10} & 8 & \frac{227}{10} \\ \frac{2}{5} & \frac{3}{5} & \frac{3}{5} & \frac{8}{5} \end{bmatrix} - \begin{bmatrix} \frac{41}{5} & \frac{19}{2} & \frac{-67}{10} & \frac{-353}{5} \\ \frac{-8}{5} & \frac{-27}{10} & 4 & \frac{247}{10} \\ \frac{7}{5} & \frac{3}{5} & \frac{8}{5} & \frac{-7}{5} \end{bmatrix}$$

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$$= \begin{bmatrix} \frac{99}{10} & 7 & \frac{-83}{10} & \frac{-997}{10} \\ \frac{-23}{5} & \frac{-9}{2} & \frac{24}{5} & \frac{473}{10} \\ \frac{19}{10} & \frac{13}{5} & \frac{37}{10} & \frac{111}{10} \end{bmatrix} - \begin{bmatrix} \frac{89}{10} & 2 & \frac{-63}{10} & \frac{-967}{10} \\ \frac{-33}{5} & \frac{-5}{2} & \frac{4}{5} & \frac{493}{10} \\ \frac{29}{10} & \frac{13}{5} & \frac{47}{10} & \frac{81}{10} \end{bmatrix}$$

$$= D - E = F - G = J - K$$

are three proper splittings of A. Here $\rho(D^\dagger E) = 0.5899 < 1$, $\rho(F^\dagger G) = 0.8378 < 1$, $\rho(J^\dagger K) = 0.8713 < 1$ but $\rho(H) = 2.1125 \nless 1$.

Three step Alternating iterative Scheme

The convergence and comparison theorem of the proposed iteration scheme which we proved as the followings:

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THEOREM (1)

Let $A = B - C = X - Y = S - T$ be three proper regular splittings of a semi-monotone matrix $A \in \mathbb{R}^{m\times n}$. Then $\rho(H) = \rho(S^{\dagger}TX^{\dagger}YB^{\dagger}C) < 1$.

THREE STEP ALTERNATING ITERATIVE SCHEME

The convergence and comparison theorem of the proposed iteration scheme which we proved as the followings:

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THEOREM (2)

Let $A \in \mathbb{R}^{m\times n}$ be a semi-monotone matrix and $A = D - E = F - G = J - K$ be three proper regular splittings of A with $R(D + J - A + KF^\dagger E) = R(A)$ and $N(D + J - A + KF^\dagger E) = N(A)$. Then, $\rho(H) \leq \min\{\rho(D^\dagger E), \rho(F^\dagger G), \rho(J^\dagger K)\} < 1$.

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$$PAx = Pb, \qquad A \in \mathbb{R}^{m\times n},\ x \in \mathbb{R}^n \ and\ b \in \mathbb{R}^m \tag{6}$$

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where the matrix P is a non singular matrix of order m. Let $PA = K_p - L_p$ be a splitting of the matrix $PA \in \mathbb{R}^{n\times n}$, where K_p and L_p have same order as of PA.

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- The iterative scheme of the modified system (6) is defined by,

$$x^{k+1} = K_p^{\dagger} L_p x^k + K_p^{\dagger} Pb. \tag{7}$$

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  If $PA = K_p - L_p$ is a proper splitting of the matrix PA, then the iterative scheme (7) will converge to the least square solution $A^\dagger b$ for any initial guess x^0 if and only if $\rho(K_p^\dagger L_p) < 1$.

Preconditioned Iterative Method

The comparison between preconditioned approach and proper weak regular splitting approach has discussed in the next theorem.

Theorem

Let $A = M - N$ be a proper regular splitting of a semi-monotone matrix $A \in \mathbb{R}^{m \times n}$. Assume that there exists an orthogonal matrix $P \in \mathbb{R}^{m \times m}$ such that $A^{\dagger} P^{-1} \geq 0$. If $PA = M_p - N_p$ is a proper weak regular splitting of PA and $M_p^{\dagger} P \geq M^{\dagger}$, then $\rho(M_p^{\dagger} N_p) \leq \rho(M^{\dagger} N) < 1$.

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Theorem

Let $A = M - N$ be a convergent proper splitting of $A \in \mathbb{R}^{m\times n}$. Let $P \in \mathbb{R}^{m\times m}$ be an nonpositive orthogonal matrix such that $PA = M_p - N_p$ is a proper regular splitting of PA. If $A^{\dagger} \leq 0$ and $M_p^{\dagger}P \leq M^{\dagger}$, then $\rho(M_p^{\dagger}N_p) \leq \rho(M^{\dagger}N) < 1$.

ITERATIVE SCHEME BASED ON REGULARIZATION

- To obtain the unique least square solution $A^{\dagger}b$ of $Ax = b$, we need to solve the normal equation $A^T Ax = A^T b$. But in general the matrix $A^T A$ is ill-conditioned matrix [4].

Iterative scheme based on Regularization

- To obtain the unique least square solution $A^{\dagger}b$ of $Ax = b$, we need to solve the normal equation $A^TAx = A^Tb$. But in general the matrix A^TA is ill-conditioned matrix [4].
- Therefore, we consider the following well-posed linear system:

$$(A^TA + \lambda I)x = A^Tb, \tag{8}$$

where $\lambda > 0$ is called regularization parameter.

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- Therefore, we consider the following well-posed linear system:

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where $\lambda > 0$ is called regularization parameter.

- In [1], it is proved that the matrix $A^T A + \lambda I$ is nonsingular for every $\lambda > 0$. If we assume $B = A^T A + \lambda I$, then the system (8) reduces to the following nonsingular system:

$$Bx = A^T b. \tag{9}$$

- In [1], it also proved that $B^{-1}A^T b$ converges to $A^{\dagger} b$ when $\lambda \to 0$.

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- We consider $B = M_\lambda - N_\lambda$ is a splitting of the nonsingular matrix B, then the iterative scheme

$$x^{k+1} = M_\lambda^{-1}N_\lambda x^k + M_\lambda^{-1}A^Tb \quad (10)$$

for the system (9) converges to $B^{-1}A^Tb$ which is equal to $A^\dagger b$.

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for the system (9) converges to $B^{-1}A^T b$ which is equal to $A^\dagger b$.

We have the following comparison result for the above setting:

Theorem

Suppose $A = M - N$ be a proper convergent weak splitting of type II. Let $B = M_\lambda - N_\lambda$ be a convergent weak splitting of type I of the matrix B. If $M_\lambda^{-1} A^T \geq M^\dagger$, then $\rho(M_\lambda^{-1} N_\lambda) \leq \rho(M^\dagger N) < 1$.

Outline

Numerical Examples

Example (1)

Consider the system $Ax = b$, where $A = \begin{bmatrix} 6.2 & 9.7 & -7.5 & -4.3 \\ 3.4 & -8.8 & 2.6 & 5.0 \\ -7.3 & -2.8 & 6.1 & 1.3 \end{bmatrix}$ and $b = (0, 1, -1)^T$. Clearly A is semi-monotone matrix since $A^{\dagger} \geq 0$.

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and $b = (0, 1, -1)^T$. Clearly A is semi-monotone matrix since $A^{\dagger} \geq 0$.Consider the following three proper regular splittings of A as

$$
\begin{aligned}
A &= \begin{bmatrix} 7.39962 & 9.96576 & -7.26446 & -3.80128 \\ 5.35168 & -7.64282 & 3.33142 & 5.80321 \\ -6.19108 & -1.23196 & 7.16198 & 1.85758 \end{bmatrix} - \begin{bmatrix} 1.19962 & 0.26576 & 0.23554 & 0.498716 \\ 1.95168 & 1.15718 & 0.73142 & 0.803207 \\ 1.10892 & 1.56804 & 1.06198 & 0.557576 \end{bmatrix} \quad \text{(Splt-1)} \\
&= \begin{bmatrix} 6.76084 & 11.3339 & -6.45554 & -3.97232 \\ 5.8871 & -8.76498 & 3.64066 & 6.46661 \\ -6.45554 & -1.09718 & 7.48736 & 1.89716 \end{bmatrix} - \begin{bmatrix} 0.56084 & 1.63388 & 1.04446 & 0.327683 \\ 2.4871 & 0.03502 & 1.04066 & 1.46661 \\ 0.84446 & 1.70282 & 1.38736 & 0.597162 \end{bmatrix} \quad \text{(Splt-2)} \\
&= \begin{bmatrix} 7.48756 & 10.7622 & -6.48554 & -3.56629 \\ 6.09602 & -8.5369 & 3.79814 & 6.56695 \\ -6.28704 & -2.2252 & 6.56918 & 1.77155 \end{bmatrix} - \begin{bmatrix} 1.28756 & 1.06216 & 1.01446 & 0.733707 \\ 2.69602 & 0.2631 & 1.19814 & 1.56695 \\ 1.01296 & 0.5748 & 0.46918 & 0.471554 \end{bmatrix} \quad \text{(Splt-3)}
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Here the spectral radius of the iteration matrix, i.e., $\rho(H) = 0.492122$ which is less than $\min\{\rho(K^\dagger L) = 0.774462, \rho(U^\dagger V) = 0.8130275, \rho(X^\dagger Y) = 0.787961\} < 1$.

Numerical Examples

Example (2)

Let $A = \begin{bmatrix} 8.3 & -6.7 & 4.0 & -2.6 \\ -7.0 & 2.9 & 0.9 & -1.3 \\ 7.7 & -3.2 & -7.4 & 3.1 \end{bmatrix} = K - L$

$$= \begin{bmatrix} -1.49724 & -13.2765 & 11.0203 & -15.1059 \\ -7.3728 & -1.87972 & -3.13564 & -4.5578 \\ 13.4498 & -3.50256 & -12.5629 & 7.12079 \end{bmatrix} - \begin{bmatrix} -9.79724 & -6.57652 & 7.02032 & -12.5059 \\ -0.3728 & -4.77972 & -4.03564 & -3.2578 \\ 5.7498 & -0.30256 & -5.16292 & 4.02079 \end{bmatrix}$$

be a convergent proper splitting since $\rho(K^{\dagger}L) = 0.861797 < 1$.

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be a convergent proper splitting since $\rho(K^{\dagger}L) = 0.861797 < 1$.

Let $P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ be an nonpositive orthogonal matrix such that $PA = K_p - L_p$

$$= \begin{bmatrix} -7.09049 & 8.31709 & -2.38373 & 4.18852 \\ -7.39842 & 3.63104 & 8.74419 & -2.92637 \\ 7.98315 & -2.80637 & 0.346307 & 1.55539 \end{bmatrix} - \begin{bmatrix} -7.09049 & 8.31709 & -2.38373 & 4.18852 \\ -7.39842 & 3.63104 & 8.74419 & -2.92637 \\ 7.98315 & -2.80637 & 0.346307 & 1.55539 \end{bmatrix}.$$

is a proper regular splitting of PA.

Numerical Examples

Example (2)

Let $A = \begin{bmatrix} 8.3 & -6.7 & 4.0 & -2.6 \\ -7.0 & 2.9 & 0.9 & -1.3 \\ 7.7 & -3.2 & -7.4 & 3.1 \end{bmatrix} = K - L$

$$= \begin{bmatrix} -1.49724 & -13.2765 & 11.0203 & -15.1059 \\ -7.3728 & -1.87972 & -3.13564 & -4.5578 \\ 13.4498 & -3.50256 & -12.5629 & 7.12079 \end{bmatrix} - \begin{bmatrix} -9.79724 & -6.57652 & 7.02032 & -12.5059 \\ -0.3728 & -4.77972 & -4.03564 & -3.2578 \\ 5.7498 & -0.30256 & -5.16292 & 4.02079 \end{bmatrix}$$

be a convergent proper splitting since $\rho(K^\dagger L) = 0.861797 < 1$.

Let $P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$ be an nonpositive orthogonal matrix such that $PA = K_p - L_p$

$$= \begin{bmatrix} -7.09049 & 8.31709 & -2.38373 & 4.18852 \\ -7.39842 & 3.63104 & 8.74419 & -2.92637 \\ 7.98315 & -2.80637 & 0.346307 & 1.55539 \end{bmatrix} - \begin{bmatrix} -7.09049 & 8.31709 & -2.38373 & 4.18852 \\ -7.39842 & 3.63104 & 8.74419 & -2.92637 \\ 7.98315 & -2.80637 & 0.346307 & 1.55539 \end{bmatrix}.$$

is a proper regular splitting of PA.
Here $A^\dagger \leq 0$, $\rho(K_p^\dagger L_p) = 0.5989862 \leq 0.8617974 = \rho(K^\dagger L) < 1$.

Numerical Examples

Table: Convergence Analysis of Alternating Scheme

Example	ϵ	N	$\|Ax_k - b\|_2$	$\|x_k - A^{\dagger}b\|_2$	MT
Ex-1	10^{-9}	8	$2.8929e^{-11}$	$6.2249e^{-12}$	0.0015
Ex-1	10^{-15}	12	$1.4315e^{-16}$	$3.0803e^{-12}$	0.0024

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Table: Comparison Analysis of Alternating Scheme

Splittings	ϵ	N	$\|Ax_k - b\|_2$	$\|x_k - A^{\dagger}b\|_2$	MT
Alternating Scheme	10^{-9}	8	$2.8929e^{-11}$	$6.2249e^{-12}$	0.0015
Splitting 1	10^{-9}	14	$1.5582e^{-9}$	$2.9480e^{-10}$	0.0026
Splitting 2	10^{-9}	22	$1.9168e^{-9}$	$4.2316e^{-10}$	0.0039
Splitting 3	10^{-9}	28	$2.9515e^{-9}$	$6.2998e^{-10}$	0.0049

Numerical Examples

Table: Comparison Analysis between $K^\dagger L$ and $K_p^\dagger L_p$

Splittings	ϵ	N	$\|Ax_k - b\|_2$	$\|x_k - A^\dagger b\|_2$	MT
$K - L$	10^{-9}	204	$2.2831e^{-7}$	$9.3700e^{-9}$	0.1361
$K_p - L_p$	10^{-9}	32	$1.1962e^{-8}$	$6.1307e^{-10}$	0.0073
$K - L$	10^{-15}	345	$2.1527e^{-13}$	$8.8351e^{-15}$	0.3721
$K_p - L_p$	10^{-15}	51	$1.4127e^{-14}$	$7.24056e^{-16}$	0.0133

Numerical Examples

Example (3)

Let us consider the following two-dimensional partial differential equation

$$-\frac{\partial^2 u}{\partial^2 x} - \frac{\partial^2 u}{\partial^2 y} + 0.5\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = f(x, y), \ (x, y) \in [0, 1] \times [0, 1]$$

with Dirichlet boundary conditions.

Numerical Examples

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with Dirichlet boundary conditions.If we use central difference scheme on a uniform grid with $(N + 2)$ nodes, then we will obtain a linear system $Ax = b$, where the coefficient matrix is of order N^2 and of the following form

Numerical Examples

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with Dirichlet boundary conditions.If we use central difference scheme on a uniform grid with $(N+2)$ nodes, then we will obtain a linear system $Ax = b$, where the coefficient matrix is of order N^2 and of the following form

$$A = I \otimes P + Q \otimes I,$$

$$P = trid\,(-(h+1), 4, (h-1)) \text{ and } Q = trid\left(-\frac{h+4}{4}, 0, \frac{h-4}{4}\right).$$

NUMERICAL EXAMPLES

The comparison analysis of the three step with the schemes of [8], and [9] are summarized in Table 6.

[8] S.Q. Shen and T. Z. Huang. Convergence and comparison theorems for double splittings of matrices. Comput. Math. Appl. 51(12):1751-1760, 2006.

[9] S. Srivastava, D. Gupta, and A. Singh. An iterative method for solving singular linear systems with index one. Afrika Matematika, 27(5-6):815-824, 2016

NUMERICAL EXAMPLES

TABLE: Comparison of error bounds and mean processing time for $\epsilon = 10^{-12}$

Order of A	Method	$\|Ax_k - b\|_2$	$\|x_k - A^{-1}b\|_2$	MT
$100 (N = 10)$	Method of [9]	$8.6544e^{-15}$	$1.5445e^{-14}$	0.00845
	Method of [8]	$3.7380e^{-12}$	$2.1915e^{-11}$	0.00138
	Three-step	$1.8618e^{-13}$	$5.2171e^{-13}$	0.00055
$400 (N = 20)$	Method of [9]	$4.3592e^{-14}$	$8.0318e^{-14}$	0.27489
	Method of [8]	$3.8949e^{-12}$	$8.2797e^{-11}$	0.37036
	Three-step	$2.0225e^{-13}$	$1.5958e^{-12}$	0.00329
$900 (N = 30)$	Method of [9]	$1.4683e^{-13}$	$3.9423e^{-13}$	3.23154
	Method of [8]	$3.9572e^{-12}$	$1.8246e^{-10}$	4.65695
	Three-step	$5.6277e^{-13}$	$3.0136e^{-12}$	0.0472
$1600 (N = 40)$	Method of [9]	$3.2206e^{-13}$	$3.0576e^{-12}$	18.58005
	Method of [8]	$4.0005e^{-12}$	$3.2221e^{-10}$	23.28531
	Three-step	$1.3450e^{-12}$	$4.1725e^{-12}$	0.19843

NUMERICAL EXAMPLES

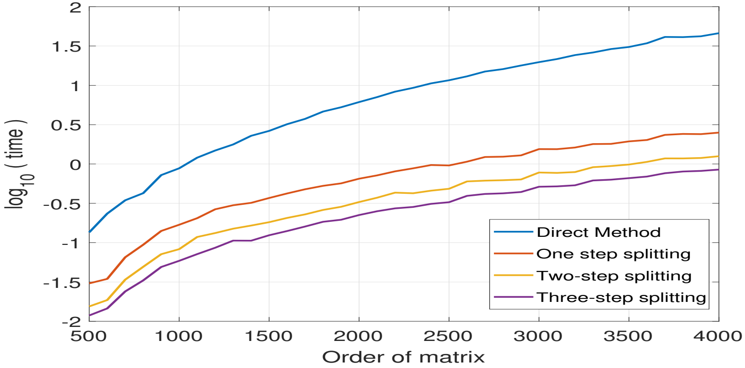

FIGURE: Comparison of mean processing time for different order matrices

Numerical Examples

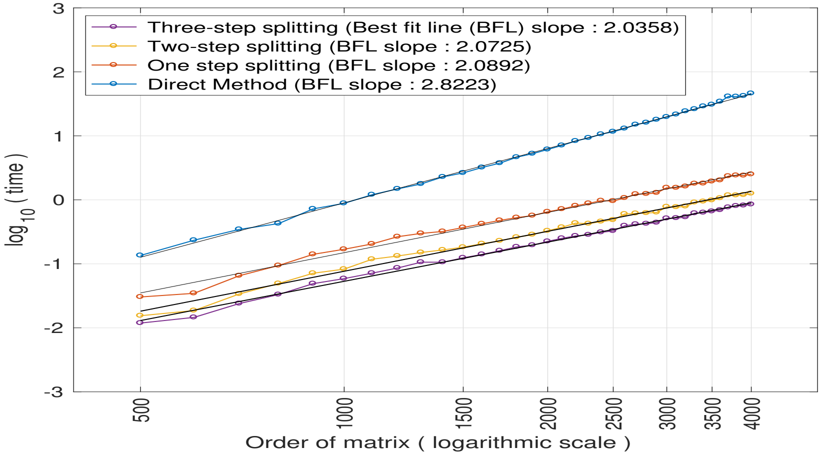

Figure: Comparison of time complexity for different order matrices

Numerical Examples

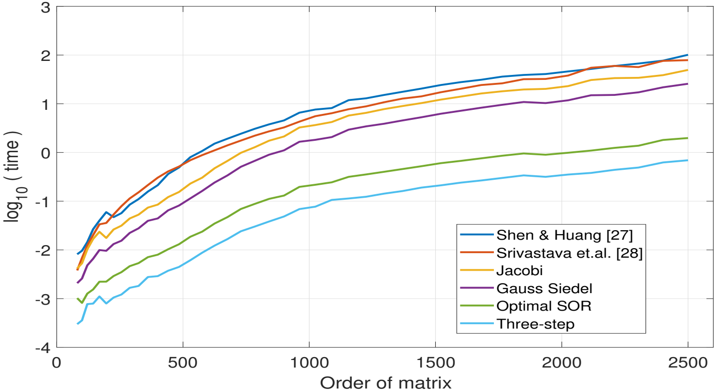

FIGURE: Comparison of mean processing time with existing methods

NUMERICAL EXAMPLES

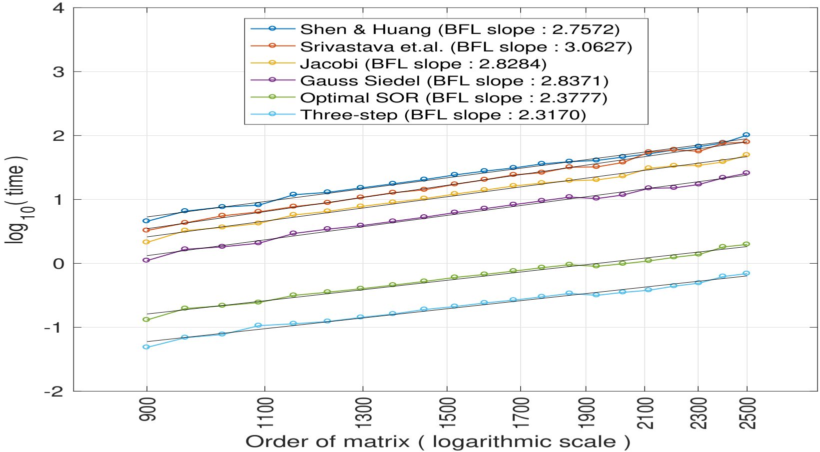

FIGURE: Comparison of time complexity with existing methods

OUTLINE

Conclusion and remarks

- We have discussed theoretical results for the proposed alternating iterative schemes. Numerical examples are provided to justify the schemes.
- The proposed scheme converges much faster than the well-known splittings. We also discuss a suitable choice of preconditioned matrix, or regularization parameter can make the system well-posed.
- Other regularization techniques can be further used for solving singular system.

OUTLINE

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Any Questions/comments ?